Assignment 3, Problem 3 [Goldenfeld Excercise 5-3 p 165] ①

$$L = \int_0^L dx \left[a t M^2 + \frac{1}{2} b M^4 + \frac{1}{2}\gamma \left(\frac{\partial M}{\partial x}\right)^2 + \frac{1}{2}\sigma \left(\frac{\partial^2 M}{\partial x^2}\right)^2 \right]$$

→ Note the problem is in one dimension.

ⓐ $M(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} e^{iq_n x} M_n$; since $M(x)$ is real, $M_n = M^*_{-n}$

Inv. Fourier transform

$$M_n = \int_0^L \frac{dx}{(2\pi)} e^{-iq_n x} M(x)$$

$$= \int_0^L \frac{dx}{(2\pi)} e^{-iq_n x} \left[\frac{1}{L} \sum_{n'=-\infty}^{\infty} M_{n'} e^{iq_{n'} x} \right]$$

Interchanging sum & integral,

$$= \frac{1}{2\pi L} \sum_{n'=-\infty}^{\infty} M_{n'} \int_0^L e^{i(q_{n'}-q_n)x} dx$$

$$= \frac{2\pi L}{2\pi L} \cdot \delta_{n,n'} M_{n'}$$

[Hence we see that the normalization for the kronecker delta is $2\pi L$.

⇒ correct expression for Kronecker δ is

$$\int_0^L e^{i(q_n - q_{n'})x} dx = 2\pi L\, \delta_{n,n'}$$

(b) Landau free energy in terms of Fourier components

$$L = \int dx\, \mathcal{L}$$

$$\mathcal{L} = a t M^2 + \frac{1}{2} b M^4 + \frac{1}{2}\gamma\left(\frac{\partial M}{\partial x}\right)^2 + \frac{1}{2}\sigma\left(\frac{\partial^2 M}{\partial x^2}\right)^2$$

$$\int dx\, a t M^2 = \int dx\, a t \frac{1}{L^2}\sum_{n,n'} e^{i(q_n + q_{n'})x} M_n M_{n'}$$

$$= \sum_{n,n'} 2\pi L \delta_{n,-n'} \frac{1}{L^2} M_n . M_{n'} \quad . a t$$

$$= \frac{2\pi}{L} a t \sum_n |M_n|^2 \quad ; \quad \text{since } M_n = M^*_{-n}$$

The fourth-order is more complicated as various momenta will couple via the Kronecker delta

$$\frac{1}{2} b \int M^4(x)\, dx = \frac{b}{2L^4}\int dx \sum_{n,n',n'',n'''} e^{i(q_n + q_{n'} + q_{n''} + q_{n'''})\cdot x} M_n M_{n'} M_{n''} M_{n'''}$$

$$= \frac{b}{2} \cdot \frac{2\pi \cdot L}{L^4} \cdot \sum_{n,n',n'',n'''} \delta_{n+n'+n''+n''',0} M_n M_{n'} M_{n''} M_{n'''}$$

$$= \frac{\pi b}{L^3} \underbrace{\sum_{n+n'+n''+n'''=0} M_n M_{n'} M_{n''} M_{n'''} =}$$

This is a difficult constrained sum to do & the advantage of going to momentum space [illegible]

Let us now consider the other two terms similarly.

<u>third term</u>

$$\int_0^L \frac{\gamma}{2}\left(\frac{\partial m}{\partial x}\right)^2 dx = \frac{\gamma}{2}\int dx \left(\frac{\partial}{\partial x}\frac{1}{L}\sum_{n=-\infty}^{\infty} m_n e^{iq_n x}\right)$$

$$\left(\frac{\partial}{\partial x}\frac{1}{L}\sum_{n'} m_{n'} e^{iq_{n'}x}\right]$$

$$= \frac{\gamma}{2}\int dx \frac{1}{L^2}\sum_{n,n'} m_n m_{n'} (iq_n)(iq_{n'}) e^{i(q_n+q_{n'})x}$$

$$= \frac{\gamma}{2} \frac{1}{L^2}\sum_{n,n'} 2\pi L\, \delta_{q_n+q_{n'},0}\, m_n m_{n'} (-1) q_n q_{n'}$$

$$= \frac{\gamma}{2} \frac{2\pi}{L}\sum_{n=-\infty}^{\infty} \cancel{|m_n|^2 q_n} \; m_n m_{-n} q_n q_{-n} * (-1)$$

now $m_n^* = m_{-n} \Rightarrow m_n m_{-n} = |m_n|^2$

& $q_n = -q_{-n}$ $\left[\text{since } q_n = \frac{2\pi n}{L}\right]$

$$= +\frac{\gamma}{2}\cdot\frac{2\pi}{L}\sum_{n=-\infty}^{\infty} |m_n|^2 q_n^2$$

& the fourth term similarly

$$\int \frac{1}{2}\sigma\left(\frac{\partial^2 M}{\partial x^2}\right)^2 dx$$

$$= \frac{\sigma}{2}\cdot\frac{2\pi}{L}\sum_{n=-\infty}^{\infty} |M_n|^2 q_n^4$$

[Please verify this]

Thus the free energy in momentum space is

$$L = \frac{2\pi}{L}\sum_{n=-\infty}^{\infty} |M_n|^2\left(at + \frac{\gamma}{2} q_n^2 + \frac{\sigma}{2} q_n^4\right)$$

$$+ \frac{2\pi}{L^3}\,\cancel{b}\sum_{n+n'+n''+n'''=0} \frac{b}{2}\cdot M_n M_{n'} M_{n''} M_{n'''}$$

(c) Minimization wrt $q_n \Rightarrow$

$$\cancel{2\gamma}\,\gamma q_n + 2\sigma q_n^3 = 0 \Rightarrow q_n = 0$$

$$\text{or} \quad \gamma + 2\sigma q_n^2 = 0$$

$$\text{or} \quad q_n^2 = -\frac{\gamma}{2\sigma}$$

So when $\gamma > 0$, the only solution is $q_n = 0$

when $\gamma < 0$, $q_n = \pm\sqrt{|\gamma|/2\sigma}$ is also a possible free energy minimum...

(c) $q_n = 0 \Rightarrow$ spatially uniform solution.
$\Rightarrow$ for $\gamma > 0$

We have the usual M^4 theory as the free energy has no additional contributions from the $\left(\frac{\partial M}{\partial x}\right)^2$ & $\left(\frac{\partial M}{\partial x^2}\right)^2$ terms.

$$\Big[\; M = 0 \text{ for } t > 0$$
$$\& \; M = \pm \sqrt{\frac{at}{2b}} \text{ for } t < 0 \;,\; \gamma > 0$$

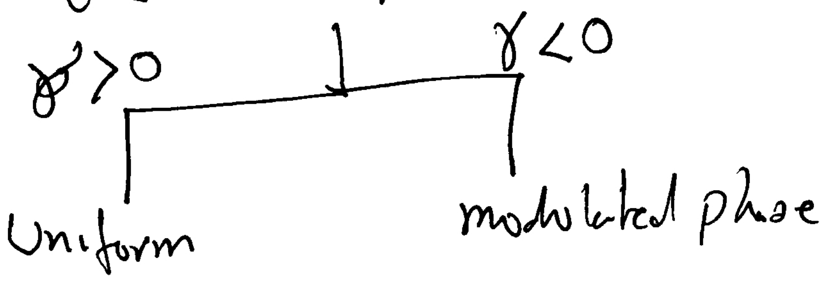
(d) The wavelength of modulation

$$|q_n| = \sqrt{\frac{|\gamma|}{2\sigma}} \;, \quad \begin{matrix} \gamma < 0 \\ t < 0 \end{matrix}$$

Note that γ could depend on another thermodynamic variable such as pressure

As the question says, we assume that only one modulation wavelength can exist. Thus we have ignored the $\sum_{n,n',n'',n'''=0} M_n \ldots M_{n'''}$ term in our discussion.

<u>To summarize</u>: At $t > 0$, $M = 0$ [Paramagnetic]
for $t < 0$ $M \neq 0$ [Ferromagnetic]



This transition to a modulated phase is known as LIFSHITZ transition

Admittedly, we did not solve this problem exhaustively. The idea of the problem was to illustrate an interesting & important situation where ~~the~~ one may observe that the ordered parameter ~~can~~ shows spatial modulation. [This is called spin density wave]. One can similarly observe a charged denity wave, or a "striped phase" in many materials undergoing a superconducting transition].

The parameter γ could in principle be tuned by another thermodynamic variable such as pressure. This transition is called the

LIFSHITZ transition & continues to fascinate condensed matter physicists.